The Coupling Coefficients of an Unsymmetrical High-Q Lossy Waveguide Resonator*

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Summary—This paper determines the minimum insertion loss and the minimum VSWR of a waveguide resonator constructed by spacing two, unequal, reactive, primarily-shunt, lossy, reflecting elements approximately one-half-wavelength apart on a lossy transmission line. The requirement that the resonant loss be small, 10 db or less, limits the size of the loss parameters and permits an approximate solution of the problem within an error of the order of $1/Q_L$. These formulas can be expressed in terms of two "coupling coefficients." Contrary to the familiar formulas derived from the low frequency analogue, however, these coupling coefficients depend, in general, on the parameters of both reflecting elements. Formulas for the loaded and unloaded Q of the resonator are derived. In general, it is not possible to determine the unloaded Q of the resonator from its loaded Q except by a limiting process. Within the order of the approximation involved, series losses cannot be distinguished from shunt losses. Accordingly they can be lumped together and one is led to the fact that a lossy admittance inverter consists of a lossless admittance inverter surrounded on both sides by series losses. This is used to justify the application of the idea of "predistortion" to the design of narrow-band, lossy, waveguide filters.

INTRODUCTION

THE RESPONSE characteristics of a lossless resonator with external coupling have been discussed in terms of Maxwell's equations and the eigenvalue solutions of the pertinent boundary value problem.[1] When loss is introduced into the problem, however, a rigorous solution is no longer available.[2]

In the customary approximation,[3] the resonator is treated in terms of its behavior at a single natural frequency where its response is that of a tuned LC circuit. To provide coupling into the resonator, the LC circuit is terminated in ideal transformers. All the loss of the resonator is associated with the tuned LC circuit while the ideal transformers, which correspond to the coupling elements of the resonator, are taken to be lossless. The final low-frequency model is pictured in Fig. 1.

For the resulting lumped-constant network, the insertion loss at resonance, P_L, is given by

$$4P_L = (1 + \beta_1 + \beta_2)^2/\beta_1\beta_2, \qquad (1)$$

while the VSWR ρ_1, measured at terminal 1, at resonance is given by

$$\rho_1 = (1 + \beta_2)/\beta_1. \qquad (2)$$

Here it should be noted that resonance occurs at the single frequency which minimizes both P_L and ρ_1. The β's are the well known coupling coefficients[3] and are given terms of the turns ratios of the ideal transformers, R_G and the loss of the LC circuit. For the network of Fig. 1, they are independent of each other.

The effect of coupling losses on the response of the low frequency analogue have been considered by Malter and Brewer[4] and by Ginzton.[5] Young[6] has considered the resonator problem in which the LC network is replaced by a lossy section of transmission line essentially as shown in Fig. 2 except that his formulas assume no losses in the coupling elements. He has shown that for a lossy, transmission-line resonator, the frequencies of minimum loss and minimum VSWR do not coincide.

In this paper, the resonator to be studied is that of Fig. 2, consisting of two lossy elements, most of whose reflection is due to the shunt discontinuity, spaced approximately an integral number of half-wavelengths apart on a uniform but lossy transmission line. The minimum insertion loss and VSWR of this resonator will be determined in terms of coupling coefficients, subject to certain limitations on the size of the loss parameters.

That some restriction is necessary, we can see as follows: Clearly a symmetric resonator ($\beta_1 = \beta_2$) whose minimum P_L and ρ are given by (1) and (2) is quasi-

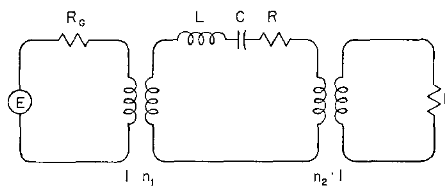

Fig. 1—Lumped constant resonator.

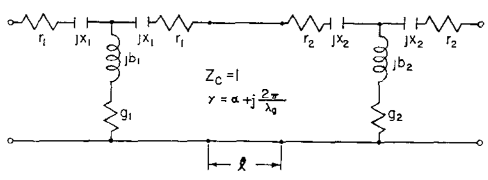

Fig. 2—Lossy transmission line resonator model.

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[1] W. W. Hansen, "A type of electrical resonator," *J. Appl. Phys.*, vol. 9, pp. 654–663; October, 1938.

[2] F. E. Borgnis and C. H. Papas, "Cavity resonators," in "Handbuch der Physik," Springer-Verlag, Berlin, Germany, vol. 16, pp. 406–422; 1958.

[3] R. Beringer, "Resonant Cavities as Microwave Circuit Elements," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 8, pp. 234–239; 1948.

[4] L. Malter and G. R. Brewer, "Microwave Q measurements in the presence of series losses," *J. Appl. Phys.*, vol. 20, pp. 918–925; October, 1949.

[5] E. L. Ginzton, "Microwave Q measurements in the presence of coupling losses," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-6, pp. 383–389; October, 1958.

[6] L. Young, "Analysis of a transmission cavity wavemeters," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-8, pp. 436–439; July, 1960.

reactive in the sense that the insertion loss of the resonator can be determined from its VSWR by eliminating $\beta_1 = \beta_2$ between (1) and (2). Now this same relation is true for the lumped-constant model regardless of loss; but, if the transmission line between two identical reflecting elements, as in Fig. 2, is sufficiently long or lossy, there will be no direct relationship between the input standing wave ratio and the insertion loss. Thus we know that for the general network of Fig. 2 no exact formulas for input VSWR and insertion loss in terms of coupling coefficients is possible.

The Coupling Coefficients

The final formulas for the coupling coefficients β_1, β_2 involve b_1, b_2, r_1, r_2, g_1, g_2, and $\mathrm{sh}(\alpha l)$ but do not contain x_1 and x_2 since their only effect is to slightly modify the line length at resonance. Here all symbols are defined in Fig. 2 and $\mathrm{sh}(\alpha l)$ is $\sinh(\alpha l)$. These formulas were obtained by the involved computations outlined in the Appendix. However they can be obtained directly, though heuristically, in the following manner.

We consider short lengths of transmission line placed on either side of the general reflecting element whose transfer[7] matrix is given by the product

$$\begin{pmatrix} 1 & 0 \\ r+jx & 1 \end{pmatrix} \begin{pmatrix} 1 & g+jb \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r+jx & 1 \end{pmatrix}. \tag{3}$$

If this transfer matrix is multiplied on both sides by the matrix

$$\begin{pmatrix} \cos\theta & j\sin\theta \\ j\sin\theta & \cos\theta \end{pmatrix} \tag{4}$$

where the line length $\theta = 2\pi l/\lambda_g$ is given by $\tan\theta = 1/b - x$ the over-all transfer matrix is given by

$$\begin{pmatrix} A_1 + jB_1 & A_2 + jB_2 \\ A_3 + jB_3 & A_4 + jB_4 \end{pmatrix}, \tag{5}$$

where

$$A_1 = a_1 \cos^2\theta - (b_2 + b_3)\sin\theta\cos\theta - a_4 \sin^2\theta$$
$$\vdots$$
$$B = b_4 \cos^2\theta + (a_2 + a_3)\sin\theta\cos\theta - b_1 \sin^2\theta, \tag{6}$$

and

$$a_1 = 1 + gr - bx; \qquad b_1 = gx + br$$
$$\vdots \qquad\qquad \vdots$$
$$a_4 = 1 + gr - b, \qquad b_4 = gx + br.$$

[7] This matrix is more commonly referred to as the *ABCD* or chain matrix. For the recent tendency to use the idea of transfer or transmission in the title see E. F. Bolinder, "Note on the matrix representation of line or two-port networks," IRE Trans. on Circuit Theory, vol. CT-4, pp. 337–339; December, 1957, and H. L. Armstrong, "Comments on the matrix representation of two-port networks," IRE Trans. on Circuit Theory (*Correspondence*), vol. CT-5, p. 147; June, 1958.

Then putting $\cos^2\theta = 1 - [1/b - x]^2 + \cdots$, $\sin\theta\cos\theta = 1/b - x + \cdots$ and $\sin^2\theta = [1/b - x]^2 + \cdots$, we obtain,

$$\begin{aligned} A_1 &= O(1/b^2); & B_1 &= br + g/b + O(1/b^3) \\ A_2 &= g + O(1/b^2) & B_2 &= b + O(1/b) \\ A_3 &= g/b^2 + O(1/b^4) & B_3 &= 1/b + O(1/b^3) \\ A_4 &= O(1/b^2) & B_4 &= br + g/b + O(1/b^3). \end{aligned} \tag{7}$$

It will be observed that the series reactance x does not appear explicitly in expressions for A and B. This follows from the assumption that the waveguide resonator is approximately an integral number of half-guide-wavelengths long. Then, x is small and we may neglect terms of the order of x/b since b is large. Actually as we shall see b is of the order of the square root of the loaded Q of the resonator. The assumption that the short, added line-lengths are lossless is implied by (4). It is justified by the fact that it gives the same result as the detailed calculation outlined in the appendix.

We have seen that some restriction must be placed on the size of the loss elements if our model is to be described by coupling coefficients. Sufficient conditions, as we shall see, are that $b^2\mathrm{sh}(\alpha l)$, $b^2 r$ and g be of the order of unity, since it is found that the expansions for the insertion loss and VSWR proceed in negative powers of b^2, so that if we neglect all terms of the size of g/b^2, $\mathrm{sh}\alpha l$, and r, a rigorous theory of the lossy resonator in terms of coupling coefficients results. On the other hand these conditions are necessary for low-loss, high-Q resonators, since P_L will be seen to contain the terms, $b^2\mathrm{sh}(\alpha l)$, $b^2 r$ and g.

When we retain only the highest order terms, we have, for the transfer matrix of the lossy reflecting element

$$\begin{pmatrix} j(br + g/b) & jb \\ j/b & j(br + g/b) \end{pmatrix}, \tag{8}$$

which reduces, for no loss, to the form of an admittance inverter.[8]

It is determined by exact calculation that the transfer matrix of the lossy transmission line between the two reflecting elements may be approximated, except for sign, by

$$\begin{pmatrix} 1 & 0 \\ j\omega + \mathrm{sh}(\alpha l) & 1 \end{pmatrix},$$

where α is the attenuation constant of the transmission line, l is an integral number of half-guide-wavelengths and ω is a suitably chosen frequency variable.

Then the transfer matrix of the lossy resonator of Fig. 2 may be written,

[8] S. B. Cohn, "Direct-coupled filters," Proc. IRE, vol. 45, pp. 187–196; February, 1957.

$$\begin{pmatrix} jb_1L_1 & jb_1 \\ j/b_1 & jb_1L_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ j\omega + \text{sh}(\alpha l) & 1 \end{pmatrix}\begin{pmatrix} jb_2L_2 & jb_2 \\ j/b_2 & jb_2L_2 \end{pmatrix}. \tag{9}$$

Here, the fact that the series and shunt losses of the reflecting elements are indistinguishable, has been used by putting $L_i = r_i + g_i/b_i^2$. When these matrices are multiplied together and terms of the order of $1/b^2$ and less are neglected, the transfer matrix becomes, except for sign,

$$\begin{pmatrix} b_1/b_2 & b_1b_2(L_1 + L_2 + \text{sh}(\alpha l) + j\omega) \\ 0 & b_2/b_1 \end{pmatrix}. \tag{10}$$

At resonance,

$$4P_L = \{b_1/b_2 + b_2/b_1 + b_1b_2[\text{sh}(\alpha l) + L_1 + L_2]\}^2. \tag{11}$$

If we define coupling coefficients β_1 and β_2 by

$$1/\beta_i = b_i^2[\text{sh}(\alpha l) + L_1 + L_2], \quad (i = 1, 2) \tag{12}$$

we find that

$$4P_L = \frac{(1 + \beta_1 + \beta_2)^2}{\beta_1\beta_2}. \tag{13}$$

On the other hand, the minimum VSWR, ρ_1, is given by the input admittance of the cascade, since it is real, by

$$\begin{aligned} \rho_1 &= b_1/b_2\{b_1/b_2 + b_1b_2[\text{sh}(\alpha l) + L_1 + L_2)\} \\ &= (1 + \beta_2)/\beta_1. \end{aligned} \tag{14}$$

Thus we see that the minimum insertion loss and input VSWR may be given in terms of coupling coefficients. These coupling coefficients, however, differ from those obtained from the lumped-constant network of Fig. 1 in that each is a function of the loss contributed by both coupling elements. Thus, in general, they are not independent parameters.

Unloaded Q

It is clear from (10) that

$$P_L(\omega) = P_L + \frac{b_1^2b_2^2\omega^2}{4}. \tag{15}$$

If we define $\Delta\omega$ as the value of ω at which $P_L(\omega) = 2P_L$, we find that

$$\Delta\omega = \pm\,[1/b_2^2 + 1/b_1^2 + L_1 + L_2 + \text{sh}(\alpha l)].$$

Thus

$$2\Delta\omega = 1/Q_L = 2[1/b_2^2 + 1/b_1^2 + L_1 + L_2 + \text{sh}(\alpha l)], \tag{16}$$

where Q_L is the loaded Q of the resonator. It is of the order of b^2 as was observed earlier. If we define Q_0 as the limit of Q_L as b_1 and b_2 approach infinity, we have

$$1/Q_0 = 2[\overline{L}_1 + \overline{L}_2 + \text{sh}(\alpha l)], \tag{17}$$

where $\overline{L}_1$ and $\overline{L}_2$ are the respective limits of L_1 and L_2. Were L_1 and L_2 independent of the b's, we would obtain the familiar relationship

$$Q_0 = (1 + \beta_1 + \beta_2)Q_L. \tag{18}$$

There is no reason, however, for assuming that L_1 and L_2 are independent of b_1 and b_2. If fact, careful measurements, discussed in a later section, indicate that for waveguide resonators L_1 and L_2 depend on b_1 and b_2. (The defining relationship $L_i = r_i + g_i/b_i^2$ tells nothing about this question since no assumptions are made about the dependence of r_i and g_i on b_i). Thus it is doubtful if the familiar relationship between Q_0 and Q_L can be used for a rigorous determination of the unloaded Q of a resonator.

The Equivalent Circuit

The transfer matrix

$$\begin{pmatrix} 1 & 0 \\ L & 1 \end{pmatrix}\begin{pmatrix} 0 & jb \\ j/b & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ L & 1 \end{pmatrix} \tag{19}$$

has precisely the form of the lossy admittance inverter (8) except for terms of the order of $1/b^3$. Thus a lossy admittance inverter is equivalent to a lossless admittance inverter surrounded on both sides by series loss. Accordingly an equivalent circuit for the lossy waveguide resonator will take the form shown in Fig. 3.

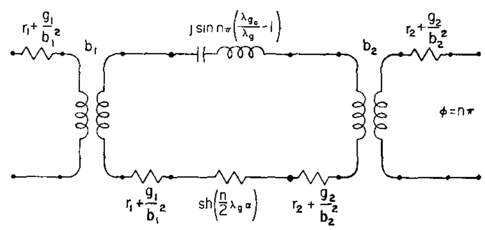

Fig. 3—Final lossy resonator schematic.

Here the symbol used by Lawson and Fano[9] for a lossless admittance inverter has been modified by omitting the values for self inductance. The series losses shown outside of the admittance inverters will ordinarily be so small as to be neglible. They are included, however, for completeness. The phase shift section has been added to give the proper phase shift at resonance. This equivalent circuit has an important advantage over that of Fig. 1, in that it gives an input impedance of the resonator which is small at the first shunt reflecting element when measured far from resonance.

Now the synthesis of high-Q waveguide filters in terms of admittance inverters and series resonant elements[10] also depends on the assumption that terms of

[9] A. W. Lawson and R. M. Fano, "The design of microwave filters," in "Microwave Transmission Circuits," M.I.T. Rad. Tab. Ser., McGraw-Hill Book Co., Inc. New York, N. Y., vol. 9, p. 662–666; 1948.

[10] H. J. Riblet, "A unified discussion of high-Q wave-guide filter design theory," IRE Trans. on Microwave Theory and Techniques, vol. MTT-6, pp. 359–368; October, 1958.

the order of $1/b^2$ may be neglected in comparison to terms of the order of unity. This follows from the fact that for the high-loss, central elements of a filter t is of the order of $\sqrt{\rho}$ which, in turn, is of the order of b.[10]

Thus the consideration of loss does not introduce any new restriction into the filter synthesis problem. It follows from (19) that a lossy direct-coupled filter will have the transfer matrix,

$$\begin{pmatrix} 0 & jb_1 \\ j/b_1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ j\omega + R_{12} & 1 \end{pmatrix}\begin{pmatrix} 0 & jb_2 \\ j/b_2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ j\omega + R_{23} & 1 \end{pmatrix}\cdots \begin{pmatrix} 0 & jb_n \\ j/b_n & 0 \end{pmatrix},$$

where $R_{ij} = \mathrm{sh}(\alpha l) + L_i + L_j$. Thus we have justified the use of predistortion[11] when applied to lossy direct-coupled, waveguide-filter design, to the extent that L_i is constant. In the above representation the series loss at each end of the filter has been neglected since it is outside of the resonant part of the structure and so has negligible effect on the performance of the filter.

An Experiment

For a resonator having a length of n half guide-wavelengths,

$$2\sqrt{P_L} = b_1/b_2 + b_2/b_1 + b_1 b_2 (L_1 + L_2 + n\,\mathrm{sh}(\alpha l)).$$

If resonators of different lengths are measured for a range of known values of b, it is possible to measure L_1, L_2 and $\mathrm{sh}(\alpha l)$. The values of $\mathrm{sh}(\alpha l)$ obtained in this way agree well with the known attenuation of waveguide. Thus the losses associated with the coupling elements can be determined with the same degree of accuracy. The following conclusions can be reached on the basis of this type of measurement. First $L \approx \mathrm{sh}(\alpha l)/2$ so that about half of the resonator loss may be attributed to the coupling elements. L_i is sufficiently constant for general filter design procedures but still varies sufficiently to result in substantial error in the determination of Q_0 if some limiting process is not used.

Conclusion

An exact model of a waveguide resonator including all elements of loss has been analyzed. The analysis is exact in the limit of High Q and proposes limits on the size of the loss elements which give the minimum insertion loss and VSWR in terms of the familiar coupling coefficients. These coupling coefficients, however, are not associated independently with the coupling elements. Moreover, the familiar formula relating the loaded Q of the resonator to its unloaded Q cannot be justified rigorously.

[11] E. G. Fubini and E. A. Guillemin, "Minimum insertion loss filters," Proc. IRE, vol. 47, pp. 37–41; January, 1959.

Appendix

In a rigorous determination of P_L and ρ, one takes the transfer matrix for the coupling element in the general form

$$\begin{pmatrix} a_1 + jb_1 & a_2 + jb_2 \\ a_3 + jb_3 & a_4 + jb_4 \end{pmatrix} \tag{20}$$

and the transfer matrix for the lossy transmission line in the form

$$\begin{pmatrix} \cosh\gamma l & \sinh\gamma l \\ \sinh\gamma l & \cosh\gamma l \end{pmatrix}. \tag{21}$$

Here $\gamma = \alpha + j\beta$, where $\beta = 2\pi l/\lambda_g$ and α is the waveguide attenuation constant. The matrix product,

$$\begin{pmatrix} a_1 + jb_1 & a_2 + jb_2 \\ a_3 + jb_3 & a_4 + jb_4 \end{pmatrix}\begin{pmatrix} \cosh\gamma l & \sinh\gamma l \\ \sinh\gamma l & \cosh\gamma l \end{pmatrix} \cdot \begin{pmatrix} a_1^1 + jb_1^1 & a_2^1 + jb_2^1 \\ a_3^1 + jb_3^1 & a_4^1 + jb_4^1 \end{pmatrix}, \tag{22}$$

is then evaluated. From it the input admittance Y of the network terminated in unity can be expressed as

$$Y = \frac{A\cos\beta + B\sin\beta + j(C\cos\beta + D\sin\beta)}{E\cos\beta + F\sin\beta + j(G\cos\beta + H\sin\beta)}. \tag{23}$$

Finally, $|\Gamma|^2$, where Γ is the reflection coefficient, takes the form

$$|\Gamma|^2 = \frac{\overline{R}\cos^2\beta + 2\overline{S}\sin\beta\cos\beta + \overline{T}\sin^2\beta}{R\cos^2\beta + 2S\sin\beta\cos\beta + T\sin^2\beta}, \tag{24}$$

while the denominator of $|\Gamma|^2$ equals $4P_L(\beta)$. To determine the minimum value of $|\Gamma|^2$ and $P_L(\beta)$, we have to differentiate the corresponding expressions with respect to β, find the corresponding resonant values of β and resubstitute. The determination of the minimum value of $P_L(\beta)$ proceeds as follows. It occurs when

$$\tan 2\beta = 2S/(R - T)$$

and on resubstitution gives a value

$$4P_L = \frac{R + T - \sqrt{(R-T)^2 + 4S^2}}{2}.$$

In the event of a symmetrical resonator, the quantity under the radical sign is a perfect square and P_L, itself, can be expressed as a perfect square. At this point, P_L is a known integral polynomial in the various parameters of the network. The principal terms occurring in it are $b^2\,\mathrm{sh}(\alpha l)$, $b^2 r$ and g, and one sees that these terms must be of the order of unity, if the resonant insertion loss is to be of the order of 10 db or less. It can also be shown that the resonant length of the resonator is independent of the waveguide loss. Moreover for a symmetrical resonator,

$$\tan\beta = 1/b - x$$

within terms of the order of $1/b^3$. It can also be readily shown that, for an unsymmetrical resonator,

$$\tan 2\beta = \frac{\tan \beta_1 + \tan \beta_2}{1 - \tan \beta_1 \tan \beta_2},$$

where β_1 and β_2 are the line lengths associated with the corresponding symmetrical resonators. This, of course, is essential if the notion of a lossy admittance inverter is to be of value.

Determining the minimum value of $|\Gamma|^2$ is considerably more involved. That it can be carried through depends on the fact that the minimum VSWR occurs at about the same frequency as the minimum value of P_L. Thus the minimum value of $|\Gamma|^2$ is obtained by dividing the minimum value of the numerator by the minimum value of the denominator. A justification of the procedure requires the values of R, S, T, $\overline{R}$, $\overline{S}$, and $\overline{T}$. These values are found to be:

$$\begin{aligned} R &= b_2^2(1 - b_1x_1)^2 + b_1^2(1 - b_2x_2)^2 \\ &\quad + 2(b_1 - b_1^2x_1)(b_2 - b_2^2x_2) + O(1) \\ S &= -b_1^2(b_2 - b_2^2x_2) - b_2^2(b_1 - b_1^2x_1) + O(b) \\ T &= b_1^2b_2^2 + O(b^2) \\ \overline{R} &= b_2^2(1 - b_1x_1)^2 + b_1^2(1 - b_2x_2)^2 \\ &\quad + 2(b_1 - b_1^2x_1)(b_2 - b_2^2x_2) + O(1) \\ \overline{S} &= -b_1^2(b_2 - b_2^2x_2) - b_2^2(b_1 - b_1^2x_1) + O(b) \\ \overline{T} &= b_1^2b_2^2 + O(b^2). \end{aligned} \tag{25}$$

Now the extreme values of both the numerator and denominator of $|\Gamma|^2$ occur when

$$\tan \beta = 1/b_1 - x_1 + 1/b_2 - x_2 + O(1/b)^3. \tag{26}$$

When this value is substituted in (24), it is found that contributions of the order of b^2 cancel out. This follows from the fact that higher order terms of the barred and unbarred quantities of (25) are the same.

Consider the rational function of t of the form,

$$\frac{a_2 + \alpha_0 + 2(b_3 + \beta_1)t + (c_4 + \gamma_2)t^2}{a_2 + \bar{\alpha}_0 + 2(b_3 + \bar{\beta}_1)t + (c_4 + \bar{\gamma}_2)t^2} \tag{27}$$

where the subscripts have been used to denote the order, in b, of the coefficient in question. For example, b_3 is of the order of b^3, while β_1 is of the order of b. It will be observed that the highest order part of each term in the numerator is the same as the highest order part of the corresponding term in the denominator. No such assumption is made concerning the remainder terms. It will be observed that this is precisely the form of $|\Gamma|^2$ in (23). Moreover, an examination of (25) will show that $b_3 = -xc_4$ and $a_2 = -xb_3$ with $x = 1/b_1 - x_1 + 1/b_2 - x_2$.

Then if (27) is differentiated with respect to t, the condition for an extreme is

$$A + Bt + Ct^2 = 0,$$

where

$$\begin{aligned} A &= a_2(\beta_1 - \bar{\beta}_1) + b_3(\bar{\alpha}_0 - \alpha_0) + \bar{\alpha}_0\beta_1 - \alpha_0\bar{\beta}_1 \\ B &= a_2(\gamma_2 - \bar{\gamma}_2) + c_4(\bar{\alpha}_0 - \alpha_0) + \bar{\alpha}_0\gamma_2 - \alpha_0\bar{\gamma}_2 \\ C &= b_3(\gamma_2 - \bar{\gamma}_2) + c_4(\bar{\beta}_1 - \beta_1) + \bar{\beta}_1\gamma_2 - \beta_1\bar{\gamma}_2 \end{aligned}$$

Putting $c = c_4$, $\beta = \beta_1 - \bar{\beta}_1$, $\alpha = \alpha_0 - \bar{\alpha}_0$ and $\gamma = \gamma_1 - \bar{\gamma}_1$, these become

$$\begin{aligned} A &= x^2c\beta + xc\alpha + O(b) \\ B &= x^2c\gamma - c\alpha + O(b) \\ C &= -xc\gamma - c\beta + O(b^3) \end{aligned}$$

and

$$t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2C}.$$

Now if we factor out the common factor c of the order b^4, we have

$$t = \frac{\alpha - x^2\gamma + O(b^{-2}) \pm \sqrt{[x^2\gamma - \alpha + O(b^{-2})]^2 - 4[x^2\beta + x\alpha + O(b^{-2})][-x\gamma - \beta + O(b^{-1})]}}{-2(x\gamma + \beta) + O(b^{-1})}.$$

Thus the significant remaining terms are the order of unity. Terms of the order of $1/b^3$ in $\tan \beta$ also contribute terms of order unity; but, as will be seen, these just cancel out. Thus errors of the order $1/b^3$ in $\tan \beta$ make no contributions to P_L or ρ. Of course, the error term in $\tan \beta$ for the two cases will differ. Thus frequencies of minimum loss and minimum VSWR are not identical. Their difference however is of the order of $1/b^3$.

The detailed calculation can be carried out in the following manner.

Since the first term under the radical sign is of the order of unity, the radical can be expanded by the binomial theorem and we may write

$$t = \frac{\alpha - x^2\gamma \pm (x^2\gamma + \alpha + 2x\beta) + O(b^{-2})}{-2(x\gamma + \beta) + O(b^{-1})}.$$

$x\gamma + \beta$ is of the order of b so that we can write

$$t = \frac{\alpha - x^2\gamma \pm (x^2\gamma + \alpha + 2x\beta)}{-2(x\gamma + \beta)}[1 + O(b^{-2})].$$

The minus sign corresponds to the minimum of $|\Gamma|^2$ and finally

$$t = x[1 + \{b^{-2}\}],$$

which is the desired result.

When this value of t is substituted in $|\Gamma|^2$, the formula for ρ previously presented results. Of course, the expression for P_L is obtained at the same time.

In the process of differentiation, we have assumed that the length of the resonator, l, is fixed and varied the frequency through the term in λ_g. In a sense, then, we have assumed that the parameters of the coupling elements are fixed with frequency. This however is not an essential assumption. It is well known[12] that the resonant frequency of a lossless waveguide resonator depends only on the length of the waveguide section and not on the frequency behavior of the coupling elements. The effect of introducing loss elements of the order of unity alters the resonant length by the order of b^{-3}. Thus as long as the loss elements vary slowly with frequency, their effect on the resonant frequency will be negligible.

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A General Power Loss Method for Attenuation of Cavities and Waveguides*

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Summary—**The usual power loss method of evaluating the damping constant and Q of cavities and the attenuation constant of waveguides, as caused by finite wall conductivity, breaks down in the case of degenerate modes and fails to predict the coupling between degenerate modes. By means of variational formulations for the lossy case it is shown how the usual power loss method may be generalized to treat the case when there are degenerate modes present. The generalized method turns out to be a particularly simple extension of the usual procedure.**

THE POWER LOSS technique has always afforded a simple and direct means of calculating the damping and attenuation constants associated with cavities and waveguides having finite wall conductivity. It should be noted, however, that an ordinary power loss analysis is not directly applicable to situations in which a degeneracy between modes is present. As Papadopoulos[1] has shown, degenerate modes are unavoidably coupled together by the surface impedance and thus a single mode approximation no longer gives a sufficient representation of the true fields in the lossy structure. A linear combination of the degenerate modes is then required in the approximation and since the coupling between these modes is not known *a priori*, the power loss technique cannot be applied. Various perturbation solutions have appeared in the literature[1–3] but these solutions fail to give a physical interpretation of the mode coupling and the degree of approximation involved.

Degeneracies are a common occurrence in a large class of geometries and therefore some simplified procedure is highly desirable. It is the purpose of this paper to generalize the usual power loss method so that it is applicable to the degenerate mode case. This generalization is obtained by using the Ritz technique in connection with variational principles for both the cavity and waveguide. The variational approach gives rise to a matrix eigenvalue problem from which all the essential information can easily be obtained. The matrix eigenvalue problems are of the greatest interest and will be presented first while the variational analyses which lead to these conclusions follow to complete the presentation. The following considerations will be limited to the most common situation in which the surface impedance is of the form

$$Z_m = R_m(1 + j),\ R_m \ll Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}},$$

although the analysis can readily be extended, treating a more general form of impedance.

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